

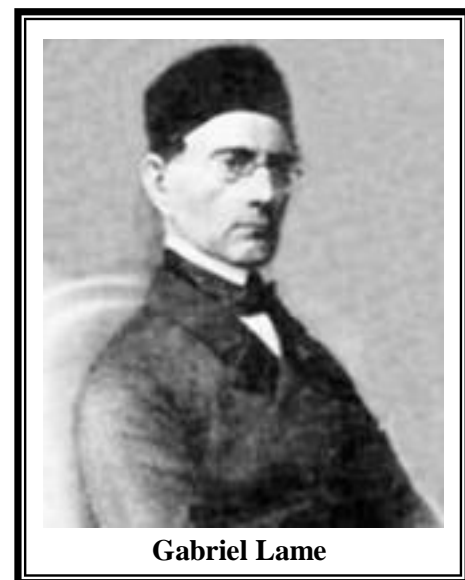
Pair of Straight Lines

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Assignment (Basic and Advance Level)

Answer Sheet of Assignment



Gabriel Lamé

The general equation of second degree

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

represents pair of straight line; if $\Delta = 0$ and $ab - h^2 \leq 0$

Clairaut (1729 A.D.) was the first to give the distance formulae although in clumsy form. He also gave the intercept form of the linear equation.

In 1818, Gabriel Lamé a civil engineer gave $mE + mE' = 0$ as the curve passing through the point of intersection of two loci $E = 0$ and $E' = 0$.



Pair of Straight Lines

3.1 Equation of Pair of Straight lines

Let the equation of two lines be

$$a'x + b'y + c' = 0 \quad \dots(i) \quad \text{and} \quad a''x + b''y + c'' = 0 \quad \dots(ii)$$

Hence $(a'x + b'y + c')(a''x + b''y + c'') = 0$ is called the joint equation of lines (i) and (ii) and conversely, if joint equation of two lines be $(a'x + b'y + c')(a''x + b''y + c'') = 0$ then their separate equation will be $a'x + b'y + c' = 0$ and $a''x + b''y + c'' = 0$.

(1) **Equation of a pair of straight lines passing through origin** : The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight line passing through the origin where a, h, b are constants.

Let the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$

$$\text{where, } m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \quad \text{and} \quad m_2 = \frac{-h - \sqrt{h^2 - ab}}{b} \quad \text{then, } m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1m_2 = \frac{a}{b}$$

Then, two straight lines represented by $ax^2 + 2hxy + by^2 = 0$ are $ax + hy + y\sqrt{h^2 - ab} = 0$ and $ax + hy - y\sqrt{h^2 - ab} = 0$.

Note : The lines are real and distinct if $h^2 - ab > 0$

The lines are real and coincident if $h^2 - ab = 0$

The lines are imaginary if $h^2 - ab < 0$

If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ should have one line common, then $(ab' - a'b)^2 = 4(ah' - a'h)(hb' - h'b)$.

The equation of the pair of straight lines passing through origin and perpendicular to the pair of straight lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by $bx^2 - 2hxy + ay^2 = 0$

If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be the square of the other, then $a^2b + ab^2 - 6abh + 8h^3 = 0$.

If the slope of one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be λ times that of the other, then $4\lambda h^2 = ab(1 + \lambda)^2$.

(2) **General equation of a pair of straight lines** : An equation of the form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g, h are constants, is said to be a general equation of second degree in x and y .

The necessary and sufficient condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(3) **Separate equations from joint equation:** The general equation of second degree be $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. To find the lines represented by this equation we proceed as follows :

Step I : Factorize the homogeneous part $ax^2 + 2hxy + by^2$ into two linear factors. Let the linear factors be $a'x + b'y$ and $a''x + b''y$.

Step II: Add constants c' and c'' in the factors obtained in step I to obtain $a'x + b'y + c'$ and $a''x + b''y + c''$. Let the lines be $a'x + b'y + c' = 0$ and $a''x + b''y + c'' = 0$.

Step III : Obtain the joint equation of the lines in step II and compare the coefficients of x , y and constant terms to obtain equations in c' and c'' .

Step IV : Solve the equations in c' and c'' to obtain the values of c' and c'' .

Step V : Substitute the values of c' and c'' in lines in step II to obtain the required lines.

Example: 1 If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product. Then c has the value

[AIEEE 2004]

- (a) -2 (b) -1 (c) 2 (d) 1

Solution: (c) We know that, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

$$\text{Given, } m_1 + m_2 = 4m_1 m_2 \Rightarrow \frac{-2c}{7} = 4\left(\frac{1}{-7}\right) \Rightarrow c = 2$$

Example: 2 If one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be $y = mx$, then

- (a) $bm^2 + 2hm + a = 0$ (b) $bm^2 + 2hm - a = 0$ (c) $am^2 + 2hm + b = 0$ (d) $bm^2 - 2hm + a = 0$

Solution: (a) Substituting the value of y in the equation $ax^2 + 2hxy + by^2 = 0$
 $\Rightarrow ax^2 + 2hx(mx) + b(mx)^2 = 0 \Rightarrow a + 2hm + bm^2 = 0$

Example: 3 If the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + K = 0$ represent two straight lines, then the value of K is [MP PET 2003]

- (a) 1 (b) 2 (c) 0 (d) 3

Solution: (b) Condition for pair of lines, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, Here $a = 12, h = -5, b = 2, g = 11/2, f = -5/2, c = K$

$$\text{Then, } 12 \times 2 \times K + 2 \times \frac{-5}{2} \times \frac{11}{2} - 12 \times \left(\frac{-5}{2}\right)^2 - 2 \times \left(\frac{11}{2}\right)^2 - K(-5)^2 = 0. \text{ On solving, we get } K = 2.$$

3.2 Angle between the Pair of Lines

(1) The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

(i) The lines are coincident if the angle between them is zero.



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\therefore Lines are coincident *i.e.*, $\theta = 0 \Rightarrow \tan \theta = 0 \Rightarrow \frac{2\sqrt{h^2 - ab}}{a + b} = 0 \Rightarrow h^2 - ab = 0 \Rightarrow h^2 = ab$

Hence, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident, *iff* $h^2 = ab$

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

$\therefore \theta = \frac{\pi}{2} \Rightarrow \cot \theta = \cot \frac{\pi}{2} \Rightarrow \cot \theta = 0 \Rightarrow \frac{a + b}{2\sqrt{h^2 - ab}} = 0 \Rightarrow a + b = 0 \Rightarrow$ coeff. of $x^2 +$ coeff. of $y^2 = 0$

Thus, the lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular *iff* $a + b = 0$ *i.e.*, coeff. of $x^2 +$ coeff. of $y^2 = 0$.

(2) The angle between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \Rightarrow \theta = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

(i) The lines are parallel if the angle between them is zero. Thus, the lines are parallel *iff*

$$\theta = 0 \Rightarrow \tan \theta = 0 \Rightarrow \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = 0 \Rightarrow h^2 = ab.$$

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel *iff* $h^2 = ab$ and $af^2 = bg^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(ii) The lines are perpendicular if the angle between them is $\pi/2$.

Thus, the lines are perpendicular *i.e.*, $\theta = \pi/2 \Rightarrow \cot \theta = 0 \Rightarrow \frac{a + b}{2\sqrt{h^2 - ab}} = 0$
 $\Rightarrow a + b = 0 \Rightarrow$ coeff. of $x^2 +$ coeff. of $y^2 = 0$

Hence, the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular *iff* $a + b = 0$

i.e., coeff. of $x^2 +$ coeff. of $y^2 = 0$.

(iii) The lines are coincident, if $g^2 = ac$.

Example: 4 The angle between the lines $x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$ is **[Karnataka CET 2003]**
 (a) 45° (b) 60° (c) 90° (d) 30°

Solution: (b) Angle between the lines is $\theta = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \tan^{-1} \left| \frac{2\sqrt{\left(\frac{-1}{2}\right)^2 - 1 \times (-6)}}{1 + (-6)} \right| = \tan^{-1} \left| \frac{2\sqrt{\frac{1}{4} + 6}}{1 + (-6)} \right| = \tan^{-1} | -1 | = \tan^{-1}(1) = \frac{\pi}{4}$,

45°

Example: 5 If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where λ is a non-negative real number, then λ is

- (a) 2 (b) 0 (c) 3 (d) 1

Solution: (a) Given that $\theta = \tan^{-1}\left(\frac{1}{3}\right) \Rightarrow \tan \theta = \frac{1}{3}$

$$\text{Now, since } \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right| \Rightarrow \frac{1}{3} = \left| \frac{2\sqrt{\left(\frac{-3}{2}\right)^2 - \lambda}}{\lambda+1} \right| \Rightarrow (\lambda+1)^2 = 9(9-4\lambda) \Rightarrow \lambda^2 + 38\lambda - 80 = 0$$

$\Rightarrow \lambda^2 + 40\lambda - 2\lambda - 80 = 0 \Rightarrow \lambda(\lambda+40) - 2(\lambda+40) = 0 \Rightarrow (\lambda-2)(\lambda+40) = 0 \Rightarrow \lambda = 2$ or -40 , but λ is a non-negative real number. Hence $\lambda = 2$.

Example: 6 The angle between the pair of straight lines represented by $2x^2 - 7xy + 3y^2 = 0$ is
[Kurukshetra CEE 2002]

- (a) 60° (b) 45° (c) $\tan^{-1}(7/6)$ (d) 30°

Solution: (b) Angle between the lines is , $\theta = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right| = \tan^{-1} \left| \frac{2\sqrt{\left(\frac{-7}{2}\right)^2 - (2)(3)}}{2+3} \right| \Rightarrow \theta = \tan^{-1} \left(\frac{2}{5} \cdot \frac{5}{2} \right) = \tan^{-1}(1) \Rightarrow$

$\theta = 45^\circ$

3.3 Bisectors of the Angles between the Lines

(1) The joint equation of the bisectors of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ (i)

$$\Rightarrow hx^2 - (a-b)xy - hy^2 = 0$$

Here, coefficient of x^2 + coefficient of $y^2 = 0$. Hence, the bisectors of the angles between the lines are perpendicular to each other. The bisector lines will pass through origin also.

Note : \square If $a = b$, the bisectors are $x^2 - y^2 = 0$ i.e., $x - y = 0, x + y = 0$

\square If $h = 0$, the bisectors are $xy = 0$ i.e., $x = 0, y = 0$.

\square If bisectors of the angles between lines represented by $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ are same, then $\frac{h}{h'} = \frac{a-b}{a'-b'}$.

\square If the equation $ax^2 + 2hxy + by^2 = 0$ has one line as the bisector of the angle between the coordinate axes, then $4h^2 = (a+b)^2$.

(2) The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by $\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$, where α, β is the point of intersection of the lines represented by the given equation.

Example: 7 The equation of the bisectors of the angles between the lines represented by $x^2 + 2xy \cot \theta + y^2 = 0$ is

- (a) $x^2 - y^2 = 0$ (b) $x^2 - y^2 = xy$ (c) $(x^2 - y^2) \cot \theta = 2xy$ (d) None of these

Solution: (a) Equation of bisectors is given by $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$ or $\frac{x^2 - y^2}{0} = \frac{xy}{\cot \theta} \Rightarrow x^2 - y^2 = 0$

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Example: 8 If the bisectors of the lines $x^2 - 2pxy - y^2 = 0$ be $x^2 - 2qxy - y^2 = 0$, then

[MP PET 1993; DCE 1999; Rajasthan PET 2003; AIEEE 2003]

- (a) $pq + 1 = 0$ (b) $pq - 1 = 0$ (c) $p + q = 0$ (d) $p - q = 0$

Solution: (a) Bisectors of the angle between the lines $x^2 - 2pxy - y^2 = 0$ is $\frac{x^2 - y^2}{xy} = \frac{1 - (-1)}{-p} \Rightarrow px^2 + 2xy - py^2 = 0$

But it is represented by $x^2 - 2qxy - y^2 = 0$. Therefore $\frac{p}{1} = \frac{2}{-2q} \Rightarrow pq = -1 \Rightarrow pq + 1 = 0$

3.4 Point of Intersection of Lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $\phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\frac{\partial \phi}{\partial x} = 2ax + 2hy + 2g = 0 \quad (\text{Keeping } y \text{ as constant})$$

and $\frac{\partial \phi}{\partial y} = 2hx + 2by + 2f = 0 \quad (\text{Keeping } x \text{ as constant})$

For point of intersection $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$

We obtain, $ax + hy + g = 0$ and $hx + by + f = 0$

On solving these equations, we get $\frac{x}{fh - bg} = \frac{y}{gh - af} = \frac{1}{ab - h^2}$ i.e. $(x, y) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right)$

Also, since $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, from first two rows

$$a \quad h \quad g \Rightarrow ax + hy + g = 0 \quad \text{and}$$

$$h \quad b \quad f \Rightarrow hx + by + f = 0 \quad \text{and then solve, we get the point of intersection.}$$

Note : \square The point of intersection of lines represented by $ax^2 + 2hxy + by^2 = 0$ is $(0, 0)$.

Example: 9 The point of intersection of the lines represented by the equation $2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$ is

- (a) $(0, 2)$ (b) $(1, 2)$ (c) $(-2, 0)$ (d) $(-2, 1)$

Solution: (c) Let $\phi \equiv 2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$

$$\frac{\partial \phi}{\partial x} = 4x + 7y + 8 = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 6y + 7x + 14 = 0$$

On solving these equations, we get $x = -2, y = 0$

Trick : If the equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The points of intersection are given by $\left\{ \frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2} \right\}$. Hence point is $(-2, 0)$

Example: 10 If the pair of straight lines $xy - x - y + 1 = 0$ and line $ax + 2y - 3 = 0$ are concurrent, then a =

- (a) -1 (b) 0 (c) 3 (d) 1

Solution: (d) Given that equation of pair of straight lines $xy - x - y + 1 = 0$

$$\Rightarrow (x-1)(y-1)=0 \Rightarrow x-1=0 \text{ or } y-1=0$$

The intersection point of $x-1=0, y-1=0$ is $(1,1)$

\therefore Lines $x-1=0, y-1=0$ and $ax+2y-3=0$ are concurrent.

\therefore The intersecting points of first two lines satisfy the third line.

Hence, $a+2-3=0 \Rightarrow a=1$

3.5 Equation of the Lines joining the Origin to the Points of Intersection of a given Line and a given Curve

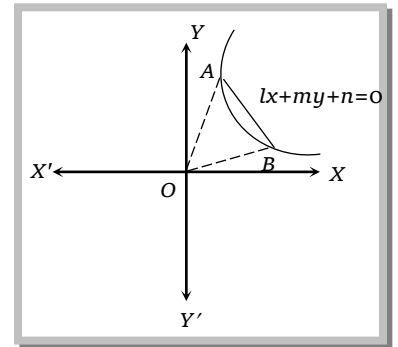
The equation of the lines which joins origin to the point of intersection of the line $lx+my+n=0$ and curve $ax^2+2hxy+by^2+2gx+2fy+c=0$, can be obtained by making the curve homogeneous with the help of line $lx+my+n=0$, which is

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$$

We have $ax^2+2hxy+by^2+2gx+2fy+c=0$ (i)

and $lx+my+n=0$ (ii)

Suppose the line (ii) intersects the curve (i) at two points A and B. We wish to find the combined equation of the straight lines OA and OB. Clearly OA and OB pass through the origin, so their joint equation is a homogeneous equation of second degree in x and y.



From equation (ii), $lx+my=-n \Rightarrow \frac{lx+my}{-n}=1$

.....(iii)

Now, consider the equation

$$ax^2+2hxy+by^2+2gx\left(\frac{lx+my}{-n}\right)+2fy\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0 \text{(i)}$$

v)

Clearly, this equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Moreover, it is satisfied by the points A and B.

Hence (iv) represents a pair of straight lines OA and OB through the origin O and the points A and B which are points of intersection of (i) and (ii).

Example: 11 The lines joining the origin to the point of intersection of the circle $x^2+y^2=3$ and the line $x+y=2$ are

- (a) $y-(3+2\sqrt{2})x=0$ (b) $x-(3+2\sqrt{2})y=0$ (c) $x-(3-2\sqrt{2})y=0$ (d) $y-(3-2\sqrt{2})x=0$

Solution: (a,b,c,d) Make homogenous the equation of circle, we get $x^2-6xy+y^2=0$

$$\Rightarrow x = \frac{6y \pm \sqrt{(36-4)y^2}}{2} = \frac{6y \pm 4\sqrt{2}y}{2} = 3y \pm 2\sqrt{2}y$$

Hence, the equation are $x=(3+2\sqrt{2})y$ and $x=(3-2\sqrt{2})y$

Also after rationalizing these equations becomes $y-(3+2\sqrt{2})x=0$ and $y-(3-2\sqrt{2})x=0$.

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Example: 12 The pair of straight lines joining the origin to the points of intersection of the line $y = 2\sqrt{2}x + c$ and the circle $x^2 + y^2 = 2$ are at right angles, if

[MP PET 1996]

(a) $c^2 - 4 = 0$ (b) $c^2 - 8 = 0$ (c) $c^2 - 9 = 0$ (d) $c^2 - 10 = 0$

Solution: (c) Pair of straight lines joining the origin to the points of intersection of the line $y = 2\sqrt{2}x + c$ and the circle $x^2 + y^2 = 2$ are

$$\Rightarrow x^2 + y^2 + (-2)\left(\frac{2\sqrt{2}x - y}{-c}\right)^2 = 0 \Rightarrow x^2 + y^2 - \frac{2}{c^2}(8x^2 + y^2 - 4\sqrt{2}xy) = 0 \Rightarrow x^2\left(1 - \frac{16}{c^2}\right) + y^2\left(1 - \frac{2}{c^2}\right) + \frac{8\sqrt{2}xy}{c^2} = 0$$

If these lines are perpendicular, $1 - \frac{16}{c^2} + 1 - \frac{2}{c^2} = 0$

$$\Rightarrow \frac{2c^2 - 18}{c^2} = 0 \Rightarrow c^2 - 9 = 0.$$

3.6 Removal of First degree Terms

Let point of intersection of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (i) is (α, β) .

$$\text{Here } (\alpha, \beta) = \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$$

For removal of first degree terms, shift the origin to (α, β) i.e., replacing x by $(X + \alpha)$ and y by $(Y + \beta)$ in (i).

Alternative Method : Direct equation after removal of first degree terms is

$$aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) = 0$$

$$\text{Where } \alpha = \frac{bg - fh}{h^2 - ab} \text{ and } \beta = \frac{af - gh}{h^2 - ab}$$

3.7 Removal of the Term xy from $f(x, y) = ax^2 + 2hxy + by^2$ without changing the Origin

Clearly, $h \neq 0$. Rotating the axes through an angle θ , we have,

$$x = X \cos \theta - Y \sin \theta \text{ and } y = X \sin \theta + Y \cos \theta$$

$$\therefore f(x, y) = ax^2 + 2hxy + by^2$$

$$\begin{aligned} \text{After rotation, new equation is } F(X, Y) &= (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta)X^2 \\ &+ 2\{(b - a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta)\}XY \\ &+ (a \sin^2 \theta - 2h \cos \theta \sin \theta + b \cos^2 \theta)Y^2 \end{aligned}$$

Now coefficient of $XY = 0$. Then we get $\cot 2\theta = \frac{a - b}{2h}$

Note : Usually, we use the formula, $\tan 2\theta = \frac{2h}{a - b}$ for finding the angle of rotation,

θ . However, if $a = b$, we use $\cot 2\theta = \frac{a - b}{2h}$ as in this case $\tan 2\theta$ is not defined.

Example: 13 The new equation of curve $12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ after removing the first degree terms

(a) $12X^2 - 7XY - 12Y^2 = 0$ (b) $12X^2 + 7XY + 12Y^2 = 0$

(c) $12X^2 + 7XY - 12Y^2 = 0$ (d) None of these

Solution: (c) Let $\phi \equiv 12x^2 + 7xy - 12y^2 - 17x - 31y - 7 = 0$ (i)

$$\therefore \frac{\partial \phi}{\partial x} \equiv 24x + 7y - 17 = 0 \text{ and } \frac{\partial \phi}{\partial y} \equiv 7x - 24y - 31 = 0$$

Their point of intersection is $(x, y) \equiv (1, -1)$

Here $\alpha = 1, \beta = -1$

Shift the origin to $(1, -1)$ then replacing $x = X + 1$ and $y = Y - 1$ in (i), the required equation is

$$12(X + 1)^2 + 7(X + 1)(Y - 1) - 12(Y - 1)^2 - 17(X + 1) - 31(Y - 1) - 7 = 0 \text{ i.e., } 12X^2 + 7XY - 12Y^2 = 0$$

Alternative Method : Here $\alpha = 1$ and $\beta = -1$ and $g = -17/2, f = -31/2, c = -7$

$$\therefore g\alpha + f\beta + c = -\frac{17}{2} \times 1 - \frac{31}{2} \times -1 - 7 = 0$$

$$\therefore \text{Removed equation is } aX^2 + 2hXY + bY^2 + (g\alpha + f\beta + c) = 0$$

$$\text{i.e., } 12X^2 + 7XY - 12Y^2 + 0 = 0 \Rightarrow 12X^2 + 7XY - 12Y^2 = 0 .$$

Example: 14 Mixed term xy is to be removed from the general equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, one should rotate the axes through an angle θ given by $\tan 2\theta =$

- (a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$ (c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

Solution: (d) Let (x', y') be the coordinates on new axes, then put $x = x' \cos \theta - y' \sin \theta, y = x' \sin \theta + y' \cos \theta$ in the equation, then the coefficient of xy in the transformed equation is 0.

$$\text{So, } 2(b-a) \sin \theta \cos \theta + 2h \cos 2\theta = 0 \Rightarrow \tan 2\theta = \frac{2h}{a-b}$$

3.8 Distance between the Pair of parallel Straight lines

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of parallel straight lines, then the distance between them is given by $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$

Example: 15 Distance between the pair of lines represented by the equation $x^2 - 6xy + 9y^2 + 3x - 9y - 4 = 0$ [Kerala (Engg.) 2000]

- (a) $\frac{15}{\sqrt{10}}$ (b) $\frac{1}{2}$ (c) $\sqrt{\frac{5}{2}}$ (d) $\frac{1}{\sqrt{10}}$

Solution: (c) The distance between the pair of straight lines given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ is } 2\sqrt{\frac{g^2 - ac}{a(a+b)}}, \text{ Here } a = 1, b = 9, c = 4, g = \frac{3}{2} = 2 \times \sqrt{\frac{9 - (-4)}{1(1+9)}} = 2 \times \sqrt{\frac{25}{10}} = \sqrt{5}$$

Example: 16 Distance between the lines represented by the equation $x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0$ is [Roorkee 1989]

- (a) $5/2$ (b) $5/4$ (c) 5 (d) 0

Solution: (a) First check for parallel lines i.e., $\frac{a}{h} = \frac{h}{b} = \frac{g}{f} \Rightarrow \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{-3}{-3\sqrt{3}}$

$$\text{which is true, hence lines are parallel. } \therefore \text{Distance between them is } 2\sqrt{\frac{g^2 - ac}{a(a+b)}} = 2\sqrt{\frac{(-3/2)^2 - 1(-4)}{1(1+3)}} = 5/2$$

3.9 Some Important Results

68 Pair of Straight Lines

(1) The lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ will be mutually perpendicular, if $g(a'+b') = g'(a+b)$.

(2) If the equation $hxy + gx + fy + c = 0$ represents a pair of straight lines, then $fg = ch$.

(3) The pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle.

(4) The area of a triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by $\frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$

(5) The lines joining the origin to the points of intersection of line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ will be mutually perpendicular, if $a^2(m^2 + 1) = 2c^2$.

(6) If the distance of two lines passing through origin from the point (x_1, y_1) is d , then the equation of lines is $(xy_1 - yx_1)^2 = d^2(x^2 + y^2)$

(7) The lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = c(bf^2 - ag^2)$

(8) The product of the perpendiculars drawn from (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is given by

$$\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$$

(9) The product of the perpendiculars drawn from origin on the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\frac{c}{\sqrt{(a-b)^2 + 4h^2}}$$

(10) If the lines represented by the general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular, then the square of distance between the point of intersection and origin is $\frac{f^2 + g^2}{h^2 + b^2}$

(11) The square of distance between the point of intersection of the lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and origin is $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$

Example: 17 The area of the triangle formed by the lines $4x^2 - 9xy - 9y^2 = 0$ and $x = 2$ is [Roorkee 2000]

- (a) 2 (b) 3 (c) $\frac{10}{3}$ (d) $\frac{20}{3}$

Solution: (c) The area of triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by

$$\left| \frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} \right|$$

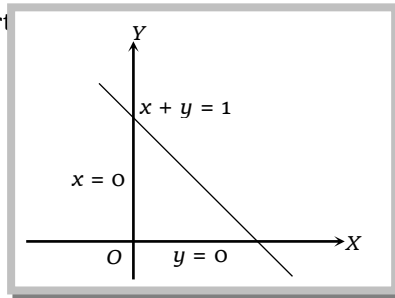
Here $a = 4, b = -9, h = -\frac{9}{2}, l = 1, m = 0, n = -2$, then area of triangle

$$= \left| \frac{(-2)^2 \sqrt{\left(\frac{-9}{2}\right)^2 - 4 \times \frac{-9}{2}}}{-9 \times (1)^2} \right| = \left| \frac{4 \sqrt{\frac{81}{4} + \frac{36}{2}}}{-9} \right| = \left| \frac{-30}{9} \right| = \frac{10}{3}$$

Example: 18 The orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$ is [IIT 1995]

- (a) $(0, 0)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{1}{4}, \frac{1}{4}\right)$

Solution: (a) Lines represented by $xy = 0$ is $x = 0, y = 0$. Then the triangle formed is right angled triangle at $O(0, 0)$, therefore $O(0, 0)$ is its orthocentre.



Example: 19 If the pair of straight lines given by $Ax^2 + 2Hxy + By^2 = 0, (H^2 > AB)$ forms an equilateral triangle with line $ax + by + c = 0$ then $(A + 3B)(3A + B)$ is [EAMCET 2003]

- (a) H^2 (b) $-H$ (c) $2H^2$ (d) $4H^2$

Solution: (d) We know that the pair of lines $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle. Hence comparing with $Ax^2 + 2Hxy + By^2 = 0$ then $A = a^2 - 3b^2, B = b^2 - 3a^2, 2H = 8ab$

Now $(A + 3B)(3A + B) = (-8a^2)(-8b^2) \Rightarrow (8ab)^2 = (2H)^2 = 4H^2$.
